# Automatic Order Reduction of Thermo-Electric Models for MEMS: Arnoldi versus Guyan


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## Abstract

*In this paper we present an automatic order reduction of a linear thermo-electric model describing a novel type of micropropulsion device. Model order reduction is essential to achieve easily to evaluate, yet accurate macromodel of the device, and is needed for simulating both the microthruster array and its driving circuitry. We present numerical simulation results of the full finite element model and the different reduced order models that describe the transient thermo-electric behaviour of the device. For the first time the advantages of an Arnoldi-algorithm-based model order reduction over a commercially available reduced order modeling after Guyan are shown.*


## 1. Introduction

A new class of high energy MEMS actuators integrates solid fuel with three silicon micromachined wafers [1]. It delivers either an impulse-bit thrust or pressure waves within a sub millimeter volume of silicon, by producing a high amount of energy from an ignitable substance contained within the microsystem. The microthruster fuel is ignited by passing an electric current through a polysilicon resistor embedded in a dielectric membrane, as shown in Fig. 1. After the ignition phase, sustained combustion takes place and forms a high-pressure, high-temperature gas mixture. Under the pressure of the gas the membrane ruptures, and an impulse is imparted to the carrier frame as the gas escapes from the tank.

The present work considers the initial heating phase of the fuel, right up to the onset of ignition, described through the following equations:

$$\nabla\bullet(\kappa\nabla T) + Q - \rho C_p\frac{\partial T}{\partial t} = 0, Q = \frac{\mathbf{j}^2}{\sigma} \qquad (1)$$

where $\kappa$ is the thermal conductivity, $C_p$ is the specific heat capacity, $\rho$ is the mass density, $T$ is the temperature distribution, Q is the heat generation, $\mathbf{j}$ is the electric current density vector and $\sigma$ is the specific electric conductivity.

We use a two dimensional axi-symmetric model, which after finite element (FE) based spatial discretization of the governing equations (1) results in a linear system of about 1000 ordinary differential equations-ODEs of the form:

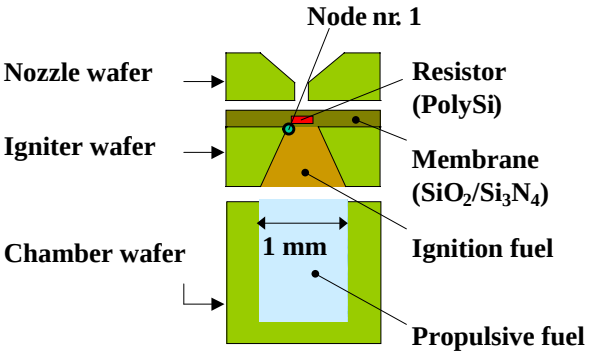



Fig. 1 Microthruster Structure.

$$[C]\{\dot{T}\} + [K]\{T\} = \{F\}I(t)^2R \qquad (2)$$

where $[K],[C] \in R^{n\times n}$ are the global heat conductivity and heat capacity matrix, $\{T(t)\},\{F\} \in R^n$ are the temperature (output) and load vector, and n is the dimension of the system. The electric current I(t) through the heater with electric resistivity R is the input to the system.

As the above number of equations n is too large for an efficient system simulation of a complete array of microthrusters and its driving circuitry, e.g., using behavioral simulators such as SABER or ELDO or a circuit simulator such as SPICE, a reduced order model:

$$[C]_r\{\dot{T}_r\} + [K]_r\{T_r\} = \{F_r\}I(t)^2R \qquad (3)$$

with the dimension r $(r \ll n)$ has been developed. The equation (3) was obtained by using two different techniques: a commercially available, reduced order modeling after Guyan [2], and an order reduction method, based on the Arnoldi algorithm [3]. The comparison of both of them is presented below.

## 2. Model Order Reduction via Guyan

The large dimension of (2) could be reduced by the elimination of internal nodes i. e., those which do not connect

to external circuitry. For steady-state problems ($[C] = 0$) it is possible to decompose the linear system into terminal and internal equations, by splitting the matrix K into four blocks:

$$\begin{bmatrix} K_{ee} & K_{ei} \\ K_{ie} & K_{ii} \end{bmatrix} \begin{Bmatrix} T_e \\ T_i \end{Bmatrix} = \begin{Bmatrix} F_e \\ F_i \end{Bmatrix} I^2(\infty)R \tag{4}$$

with the index sets e and i ranging over all external and internal nodes respectively. It is now possible to eliminate the equations for the non-terminal nodes by means of linear algebra operations (e.g., the Schur complement) [4], to get the heat conductivity matrix and the load vector of the reduced system:

$$\begin{aligned} [K]_r &= [K_{ee}] - [K_{ei}][K_{ii}]^{-1}[K_{ie}] \\ \{F_r\} &= \{F_e\} - [K_{ei}][K_{ii}]^{-1}\{F_i\} \end{aligned} \tag{5}$$

The commercial FE solver ANSYS offers the possibility of reduced order modeling also for the transient problems of the form (2). The computation of $[K]_r$ and $\{F_r\}$ is done as in (5) and the reduced heat capacity matrix is given through:

$$\begin{aligned} [C]_r = [C_{ee}] - [K_{ei}][K_{ii}]^{-1}[C_{ie}] - [C_{ei}][K_{ii}]^{-1}[K_{ie}] \\ + [K_{ei}][K_{ii}]^{-1}[C_{ii}][K_{ii}]^{-1}[K_{ie}] \end{aligned} \tag{6}$$

which is analog to the computation of the reduced mass matrix for structural dynamics, as proposed by Guyan [5]. It is now possible to expand the terminal degree of freedom (DOF) values to gain the complete temperature distribution of the device using again the equation (4) for the steady-state:

$$\{T_i\} = [K_{ii}]^{-1}\{F_i\} - [K_{ii}]^{-1}[K_{ie}]\{T_e\} \tag{7}$$

## 3. Model Order Reduction via Arnoldi

The basic idea behind the Arnoldi order reduction algorithm is to write down the algebraic relation between the input and the output of the linear system (2) in the frequency domain using a Taylor series in the Laplace variable $s = 0$:

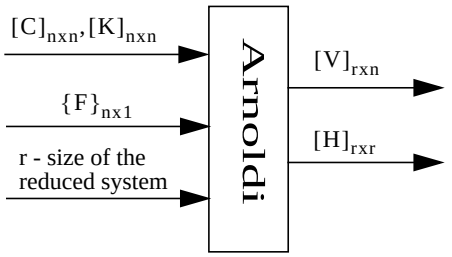
$$\{G(s)\} = -\sum_{i=0}^{\infty} \{m\}_i s^i \tag{8}$$

where $\{m\}_i = E^T(-C^{-1}K)^{-(i+1)}C^{-1}F$ is called the i*th* moment, and then to find a much lower order system whose transfer function $\{G_r(s)\}$ has the same moments as $\{G(s)\}$ up to the degree r.

The explicit computation of the moments is however circumvented. Instead, a Krylov subspace of the dimension r:

$$\begin{aligned} K_r\{A, \mathbf{b}\} &= \text{span}\{\mathbf{b}, A^2\mathbf{b}, \ldots, A^{r-1}\mathbf{b}\} \\ \text{with } A &= -[K]^{-1}[C], \text{ and } \mathbf{b} = -[K]^{-1}\{F\} \end{aligned} \tag{9}$$

is used, and through the computation of an orthogonal basis for this subspace, the matrices $[C]_r$ and $[K]_r$, and the load vector $\{F_r\}$ of the reduced system are computed. All the inputs and outputs of the Arnoldi algorithm are shown in Fig. 2.




$$[C]_r = H = V^T(-K^{-1}C)V,\ [K]_r = -I,\ \{F_r\} = \left\|-K^{-1}F\right\| \hat{e}_1$$

Fig. 2 Model reduction by Arnoldi process.

The orthogonal basis for the Krylov subspace (9) is computed iteratively and is preserved within a matrix [V]. [H] is an upper Heisenberg matrix. The property of the Krylov subspace (9) is such that the first r moments of $\{G_r(s)\}$ and $\{G(s)\}$ match, which is exactly what we want.

## 4. Results

An equation system of 1071 ODEs was reduced to 20 ODEs using each of the two presented algorithms (Fig. 3). Master degrees of freedom (external nodes) needed for Guyan algorithm, were chosen automatically by ANSYS5.7. A maximal relative error by the Arnoldi-based reduction was 0.5%, whereas this error by Guyan-based reduced order modeling ascended to over 64%.

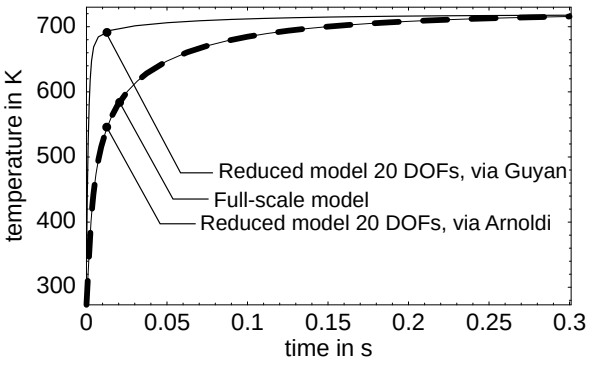



Fig. 3 Solution of the full system (dashed) and of the 20*th* order reduced system for a single node (node 1 in Fig. 1).

A large error by the Guyan algorithm based reduction is mostly due to the transient heating phase, and vanishes within the steady-state response according to equations

(4) and (5). The approximation error for the reduced heat capacity matrix (6) decreases as the order r of the reduced system grows (Fig. 4). A maximal relative error between the full-scale solution and the reduced solution of the order 200 still amounts to 6%.

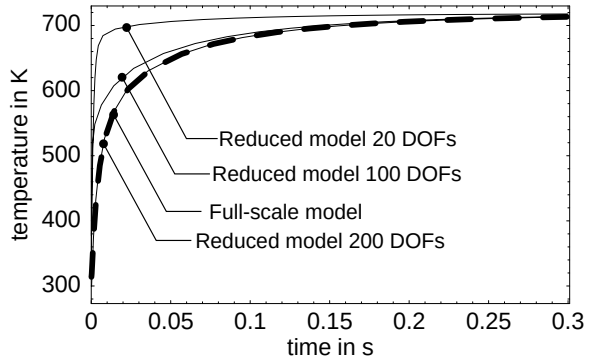



Fig. 4 Solution of the full system (dashed) and of the different order reduced systems for a single node (node 1 in Fig. 1), using Guyan reduction.

The transfer function of the full and reduced order 20 model (computed via Arnoldi) is shown in Fig. 5. Excellent agreement between both of them in the low-frequency domain (f < 10 kHz) corresponds to moment matching of order 20 (provided by the reduction algorithm) and is sufficient for the microthruster device.

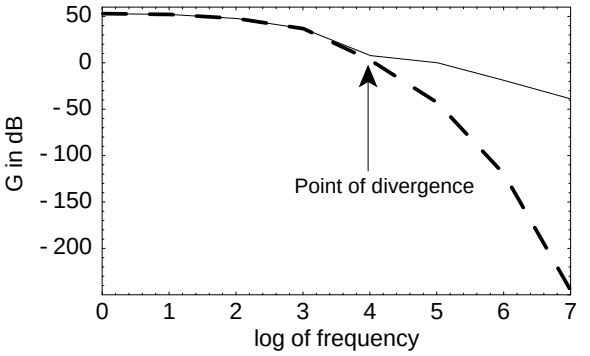



Fig. 5 Single node transfer function of the full system (dashed) and of the reduced system (full) corresponding to the node 1 in Fig. 1 for Arnoldi-based reduction.

For this case of the thermo-electric model, a simple Single-Input-Single-Output (SISO) original setup for the Arnoldi algorithm was sufficient to approximate not only a single output response (Fig. 3) but also the transient thermal response in all other nodes of the microthruster. Fig. 6 shows that a maximal mean relative difference for all the nodes between the full-scale and the recovered model (out of the reduced order 20 model) amounts only 0.14%. Hence it was possible, after the simulation of the reduced order 20 model, to recover the solution for all 1071 nodes.

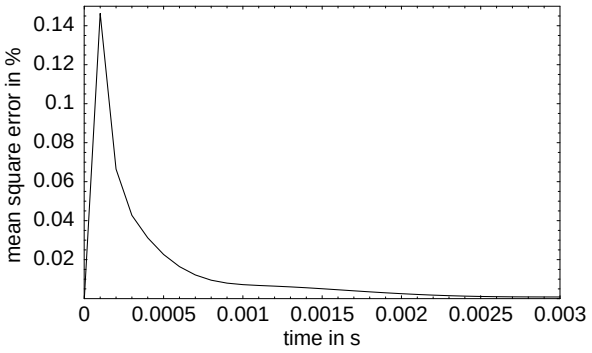



Fig. 6 Mean relative difference for all the nodes during the initial 3ms, for Arnoldi-based reduction from 1071 to 20.

## 5. Discussion

Both algorithms presented offer the possibility for the automatic order reduction of ODE systems. It has been shown though that for transient thermo-electric problems, the Arnoldi-based reduction boasts a much smaller error for the same order of the reduced system than the Guyan method. This is due to the fact that reduced order modeling after Guyan makes an attempt to generalize equation (4) for a steady-state response to the transient problem using a coordinate transformation $\{T\} = [V] \cdot \{T_r\}$ or:

$$\begin{Bmatrix} T_e \\ T_i \end{Bmatrix} = \begin{bmatrix} I \\ -K_{ii}^{-1} K_{ie} \end{bmatrix} \{T_e\} \qquad (10)$$

This leads to exact matrix condensation for the heat conductivity matrix, but an approximated condensation for the heat capacity matrix.
Arnoldi-based reduction starts with moment matching for the transient problem as it is, and also amounts to a coordinate transformation of the form:

$$\{T\} = [V] \cdot \{T_r\} + \boldsymbol{\varepsilon} \qquad (11)$$

where $\varepsilon$ is a transformation error, and $[V] \in R^{nxr}$ is gained directly as output of Arnoldi algorithm. It has been shown that for our case of the thermo-electric model, by reduction from 1071 to 20 counts $\varepsilon \cong 0$.

A further big advantage of the Arnoldi algorithm is its iterative nature: whereas the Guyan algorithm demands expensive matrix operations for the equations (5) and (6), only vector matrix multiplications are needed in each iterative step of the Arnoldi orthogonalisation procedure. Finally, the fact that no master degrees of freedom need to be chosen by the Arnoldi algorithm contributes to its convenience.
Currently, a software package is being developed which generates linear reduced-order models directly from ANSYS data files that contain more than 10 000 degrees of freedom. From the three dimensional geometry and knowledge of the governing PDE of the model it forms a netlist suitable for use in the behavioral simulator SABER (Fig. 7).

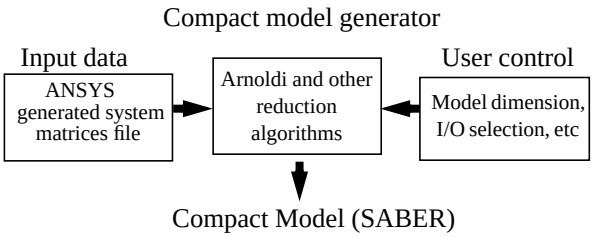



Fig. 7 Software block diagram.

## 6. Acknowledgments

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